New Facts about Berezinians*

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Abstract

We consider a new formula for Berezinian (superdeterminant). The Berezinian of a supermatrix A is expressed as the ratio of polynomial invariants of A. This formula follows from recurrence relations existing for supertraces of exterior powers.

Tools of supermathematics have become an essential part of the mathematical baggage of theoretical physics. On the other hand, still there are many important questions corresponding to statements well-known in the ordinary case, answers to which are not at all clear in the supercase.

Here we discuss some of these problems. We consider deep relations that arise in the supercase between Berezinian (superdeterminant) and exterior powers. In particular, this allows to give a new expression for the Berezinian in terms of polynomial invariants of a matrix. Details see in our paper [3].

Recall the relations between traces and (ordinary) determinant. If A is a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A = ad - bc = \frac{1}{2}\left((a+d)^2 - (a^2+2bc+d^2)\right)$ $= \frac{1}{2}\left(\mathrm{Tr}^{\,2} A - \mathrm{Tr}\, A^2\right)$ and $\det(1+Az) = 1 + z\mathrm{Tr}\, A + z^2 \det A$. This is a commonplace. In general, if A is an $n\times n$ matrix, then one can consider the following polynomial of degree n:

$$R_A(z) = \det(1+Az) = \sum_{k=0}^{n} c_k(A) z^k\,, \tag{1}$$

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the characteristic polynomial of the matrix A. (For our purposes it is more convenient to consider the above polynomial instead of $\det(A-z)$.) One can easily calculate the coefficients of this polynomial by taking the derivative with respect to z. We arrive at the relation:

$$\frac{d}{dz}\det(1+Az) = \sum_{k=1}^{n} kc_k(A)z^{k-1} =$$
$$\det(1+Az)\mathrm{Tr}\,\left((1+Az)^{-1}A\right) = \sum_{k=0}^{n} c_k(A)z^k \sum_{k=0}^{\infty}(-1)^k s_{k+1}z^k\,,$$

where we denoted $s_k(A) = \mathrm{Tr}\,A^k$. This leads to recurrence relations expressing $c_k(A)$ in terms of $s_k(A)$:

$$c_0 = 1,\ c_1 = s_1,\ \ldots,\ c_{k+1} = \frac{1}{k+1}(s_1c_k - s_2c_{k-1} + \ldots + (-1)^k s_{k+1}),\ \ldots \tag{2}$$

In particular,

$$\det A = c_n(A) \quad \text{if } A \text{ is an operator on an } n\text{-dimensional space,} \tag{3}$$

and it can be expressed via $s_k(A)$. These are standard facts in linear algebra.

What about a generalisation of the above formulae to the supercase?

Let V be a $p|q$-dimensional superspace. One can describe it in the following way. Let $V_0 \oplus V_1$ be the direct sum of p-dimensional and q-dimensional vector spaces. Let $\{\mathbf{e}_i\}$ $(i = 1,\ldots,p)$ and $\{\mathbf{f}_\alpha\}$ $(\alpha = 1,\ldots,q)$ be bases in the spaces V_0, V_1, respectively. Consider linear combinations $\sum_{i=1}^p a^i\mathbf{e}_i + \sum_{\alpha=1}^q b^\alpha\mathbf{f}_\alpha$ where coefficients a^i are even elements of some Grassmann algebra Λ and b^α are odd elements of this Grassmann algebra. Such linear combinations are considered as points of the $p|q$-dimensional superspace V.

Let A be an even linear operator on this space. The (super)matrix of the operator A has the form $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11}\end{pmatrix}$, where A_{00}, A_{11} are $p\times p$ and $q\times q$ matrices, respectively, with even entries taken from the Grassmann algebra Λ, and A_{01}, A_{10} are $p\times q$ and $q\times p$ matrices, respectively, with odd entries from the Grassmann algebra Λ. Such a (super)matrix is called even.

The Berezinian (superdeterminant) of an even matrix A is given by the famous formula due to F. A. Berezin (see [1]):

$$\mathrm{Ber}\,A = \frac{\det\left(A_{00} - A_{01}A_{11}^{-1}A_{10}\right)}{\det A_{11}}\,. \tag{4}$$

Berezinian is a multiplicative function of matrices, $\mathrm{Ber}\,(AB) = \mathrm{Ber}\,A \cdot \mathrm{Ber}\,B$. Hence Ber is well-defined on operators. Berezinian is related with supertrace in the same way as the ordinary determinant, with trace: for an even supermatrix D

$$\mathrm{Ber}\, e^D = e^{\mathrm{Tr}\, D}\,. \tag{5}$$

We denote the supertrace of a supermatrix by the same symbol as the trace of an ordinary matrix. Recall that for an even supermatrix

$$\mathrm{Tr}\, D = \mathrm{Tr}\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} = \mathrm{Tr}\, D_{00} - \mathrm{Tr}\, D_{11}\,.$$

Instead of the characteristic polynomial (1) one has to consider the *characteristic rational function* $R_A(z) = \mathrm{Ber}\,(1 + Az)$. We note that the straightforward use of expression (4) for the analysis of the characteristic function leads to a confusion.

Let us step back and consider the geometrical meaning of coefficients $c_k(A)$ in formula (1) for the ordinary case. Suppose that $\{\mathbf{e}_i\}$ is an eigenbasis of a linear operator A on an n-dimensional space V: $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$ $(i = 1, \dots, n)$. Then

$$R_A(z) = \det(1{+}Az) = \prod_{i=1}^{n}(1{+}\lambda_i z) = \sum_{k=0}^{n}\left(\prod_{j_1<j_2<\dots<j_k}\lambda_{j_1}\dots\lambda_{j_k}\right) z^k = \sum_{k=0}^{n} c_k(A) z^k\,.$$

Consider the basis consisting of wedge products $\{\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_k}\}$ $(1 \le j_1 < j_2 < \dots < j_k \le n)$ in the exterior power $\wedge^k V$. Then $\lambda_{j_1}\dots\lambda_{j_k}$ is the eigenvalue corresponding to the basis vector $\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_k}$. Hence we see that for the polynomial $\det(1 + Az)$,

$$c_k(A) = \mathrm{Tr}\,\wedge^k A,$$

where we denote by $\wedge^k A$ the operator induced by A in the exterior power $\wedge^k V$.

This formula can be straightforwardly generalised to the supercase (see [4], [3]). Suppose that $\{\mathbf{e}_i, \mathbf{f}_\alpha\}$ is an eigenbasis of a linear operator A in a $p|q$-dimensional superspace V: $A\mathbf{e}_i = \lambda_i\mathbf{e}_i$, $A\mathbf{f}_\alpha = \mu_\alpha\mathbf{f}_\alpha$. Here $\{\mathbf{e}_i\}$, $i = 1, \dots, p$, are even eigenvectors and $\{\mathbf{f}_\alpha\}$, $\alpha = 1, \dots, q$, are odd eigenvectors. Then

$$R_A(z) = \mathrm{Ber}\,(1 + Az) = \sum_{k=0}^{\infty} c_k(A) z^k = \prod_{i=1,\alpha=1}^{i=p,\alpha=q}\frac{1+\lambda_i z}{1+\mu_\alpha z} =$$
$$\sum_{r=0}^{p}\sum_{s=0}^{\infty}\left(\prod_{j_1<j_2<\dots<j_r}\lambda_{j_1}\dots\lambda_{j_r}\right) z^r \left(\prod_{\beta_1\le\beta_2\le\dots\le\beta_s}(-1)^s\mu_{\beta_1}\mu_{\beta_2}\dots\mu_{\beta_s}\right) z^s\,. \tag{6}$$

Consider the basis $\{\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_r}\wedge\mathbf{f}_{\beta_1}\wedge\dots\wedge\mathbf{f}_{\beta_s}\}$ $(1\leq j_1<j_2<\dots<j_k\leq p, 1\leq\beta_1\leq\dots\leq\beta_s\leq q,\ r+s=k)$ in the exterior power $\wedge^k V$. Then $\lambda_{j_1}\dots\lambda_{j_r}\mu_{\beta_1}\dots\mu_{\beta_s}$ is the eigenvalue corresponding to the basis vector $\mathbf{e}_{j_1}\wedge\dots\wedge\mathbf{e}_{j_r}\wedge\mathbf{f}_{\beta_1}\wedge\dots\wedge\mathbf{f}_{\beta_s}$. Hence in the same way as above the coefficients $c_k(A)$ of the expansion of the characteristic function at zero give traces of the exterior powers:

$$R_A(z)=\mathrm{Ber}\,(1+Az)=\sum_{k=0}^{\infty}c_k(A)z^k,\ \text{ where } c_k(A)=\mathrm{Tr}\wedge^k A\ (k=0,1,2,\dots)\,. \tag{7}$$

Relations (2) between $c_k(A)$ and $s_k(A)=\mathrm{Tr}\,A^k$ remain the same as in the ordinary case because of (5). The essential difference is that now $R_A(z)$ is a fraction, not a polynomial as in (1); there are infinitely many terms $c_k(A)$ in the power expansion (7).

Consider now the expansion of the characteristic function $R_A(z)$ at infinity. It leads to traces of the exterior powers of the inverse matrix. Indeed, $\mathrm{Ber}\,(1+Az)=z^{p-q}\mathrm{Ber}\,A\cdot\mathrm{Ber}\,(1+A^{-1}z^{-1})$. From (7) it follows that

$$R_A(z)=z^{p-q}\mathrm{Ber}\,A\cdot\mathrm{Ber}\,(1+A^{-1}z^{-1})=z^{p-q}\mathrm{Ber}\,A\sum_{k=0}^{\infty}c_k(A^{-1})z^{-k}=$$
$$\sum_{k\leq p-q}\left(\mathrm{Ber}\,A\cdot c_{p-q-k}(A^{-1})\right)z^k=\sum_{k\leq p-q}c^*_k(A)z^k \tag{8}$$

near infinity, where we have denoted by

$$c^*_k(A)=\mathrm{Ber}\,A\cdot c_{p-q-k}(A^{-1})=\mathrm{Ber}\,A\cdot\mathrm{Tr}\wedge^{p-q-k}A^{-1},\ (k=p-q,\ p-q-1,\dots)\,. \tag{9}$$

The coefficient $c^*_k(A)$ can be interpreted as the trace of the representation on the space $\mathrm{Ber}\,V\otimes\wedge^{p-q-k}V^*$.

In the ordinary case when V is an n-dimensional vector space so that $p=n, q=0$, then both (7) and (8) are the same polynomial. Comparing them, we see that

$$c_k(A)=\mathrm{Tr}\,\wedge^k A=c^*_k(A)=\det A\cdot\mathrm{Tr}\,\wedge^{n-k}A^{-1}\,. \tag{10}$$

This is a well-known identity between minors of the matrix A and its inverse A^{-1}. In particular, for $k=n$ we arrive at (3). Relation (10) holds for any invertible operator A. This is due to a canonical isomorphism existing in the ordinary case between the spaces $\wedge^k V$ and $\det V\otimes\wedge^{n-k}V^*$:

$$\wedge^k V\approx\det V\otimes\wedge^{n-k}V^*\,. \tag{11}$$

What happens in the supercase? Both expansions (7) and (8) are infinite series. Claim: the coefficients of both series form **recurrent sequences**. Indeed, we see from (6) that the function $R_A(z)$ is the ratio of two polynomials of degrees p and q, respectively:

$$R_A(z) = \mathrm{Ber}\,(1 + Az) = \frac{P(z)}{Q(z)} = \frac{1 + a_1 z + a_2 z^2 + \cdots + a_p z^p}{1 + b_1 z + b_2 z^2 + \cdots + b_q z^q}\,. \tag{12}$$

Comparing this fraction with the expansion of $R_A(z)$ around zero we arrive at the recurrence relations

$$c_{k+q} + b_1 c_{k+q-1} + \cdots + b_q c_k = 0 \tag{13}$$

satisfied for all $k > p-q$. Comparing the fraction in (12) with the expansion of $R_A(z)$ around infinity we again arrive at recurrence relations:

$$c_k^* + b_1 c_{k-1}^* + \cdots + b_q c_{k-q}^* = 0$$

satisfied for all $k < 0$. We see that both sequences $\{c_k(A)\}$ and $\{c_k^*(A)\}$ satisfy the same recurrence relations of order q. It is convenient to consider these sequences for all integer k by setting $c_k = 0$ for all $k < 0$ and $c_k^* = 0$ for all $k > p-q$. Combine these two sequences in one sequence by considering the differences:

$$\gamma_k = c_k - c_k^*\,.$$

The sequence $\{\gamma_k\}$ satisfies the same recurrence relations **for all integer** k:

$$\gamma_k + b_1 \gamma_{k-1} + \cdots + b_q \gamma_{k-q} = 0, \quad \text{for all } k\,.$$

Note that in this formula the terms $c_k = \mathrm{Tr}\,\wedge^k A$ and $c_k^* = \mathrm{Ber}\,A \cdot \mathrm{Tr}^{\,p-q-k} A^{-1}$ are simultaneously non-zero only in a finite range where $k = 0, 1, \ldots, p-q$. Otherwise $\gamma_k = c_k - c_k^*$ equals either $c_k(A)$ for $k > p-q$ or $-c_k^*$ for $k < 0$.

The condition that $\{\gamma_k\}$ is a recurrent sequence of order q can be rewritten in the following closed form:

$$\det \begin{pmatrix} \gamma_k & \cdots & \gamma_{k+q} \\ \cdots & \cdots & \cdots \\ \gamma_{k+q} & \cdots & \gamma_{k+2q} \end{pmatrix} = 0 \quad \text{for all } k \in \mathbb{Z}\,. \tag{14}$$

Using relations (13) for c_k only, one can reconstruct the function $R_A(z)$ and all rational invariants of the matrix A, including $\mathrm{Ber}\,A$, via the first $p+q$ traces $c_k = \mathrm{Tr}\,\wedge^k A$ $(k = 1, 2, \ldots, p+q)$, by a recursive procedure. However, equation (14) for the differences $\gamma_k = c_k - c_k^*$ gives much more.

Formula (14) stands in the supercase instead of the equality (10) holding in the ordinary case. This leads to highly non-trivial relations between exterior powers $\wedge^k V$ and $\mathrm{Ber}\, V \otimes \wedge^{p-q-k} V^*$ instead of the canonical isomorphism (11).

Formula (14) also gives a closed expression for $\mathrm{Ber}\, A$ in terms of traces. Indeed, it follows from (7)–(9) that

$$\mathrm{Ber}\, A = \mathrm{Tr}\, \wedge^{p-q} A - \gamma_{p-q}\,. \tag{15}$$

(In the ordinary case $q = 0$, $\gamma_{p-q} = 0$ we arrive at (3).) Now, by considering relation (15) and identity (14) for $k = p - q$ we arrive at the formula

$$\mathrm{Ber}\, A = \frac{\det \begin{pmatrix} c_{p-q}(A) & \dots & c_p(A) \\ \dots & \dots & \dots \\ c_p(A) & \dots & c_{p+q}(A) \end{pmatrix}}{\det \begin{pmatrix} c_{p-q+2}(A) & \dots & c_{p+1}(A) \\ \dots & \dots & \dots \\ c_{p+1}(A) & \dots & c_{p+q}(A) \end{pmatrix}}\,. \tag{16}$$

Here as before we set $c_k = 0$ for $k < 0$.

For example, let A be an even operator in a $p|1$-dimensional vector space. Then

$$\mathrm{Ber}\, A = \frac{\det \begin{pmatrix} c_{p-1}(A) & c_p(A) \\ c_p(A) & c_{p+1}(A) \end{pmatrix}}{c_{p+1}(A)} = c_{p-1}(A) - \frac{c_p^2(A)}{c_{p+1}(A)}\,.$$

The rational expression in (16) is essentially different from the original formula (4), where the numerator and denominator are not invariant functions of the matrix A. Compared to it, the numerator and denominator of the fraction in formula (16) are invariant polynomials.

One can show that these invariant polynomials are the traces of the representations corresponding to certain Young diagrams. Namely, the numerator in (16) is equal to the trace of the action of the operator A on an invariant subspace in the space of tensors $V^{\otimes N}$ corresponding to the rectangular Young diagram $D_{p,q+1}$ with p rows of length $q+1$. Respectively, the denominator in (16) is equal to the trace of the action of the operator A on an invariant subspace corresponding to the Young diagram $D_{p+1,q}$ [1]. This follows from the well-known *Schur–Weyl formula* [5] which can be generalised to the supercase (see e.g. in [2])

[1]If A is an ordinary $p \times p$ matrix, then by (3), $\det A = c_p(A)$ simply equals the trace $\mathrm{Tr}\, \wedge^p A$ on the one-dimensional space of totally antisymmetric p-tensors, corresponding to the Young diagram $D_{p,1}$ with p rows of length 1.

Denote the invariant polynomials in the numerator and denominator of the fraction in (16) by $\operatorname{Ber}^+(A)$ and $\operatorname{Ber}^-(A)$, respectively. What is the meaning of $\operatorname{Ber}^+(A), \operatorname{Ber}^-(A)$ in terms of the eigenvalues of the operator A?

Compare (16) with expression (12) for the characteristic function $R_A(z)$. Let $\{\lambda_1, \ldots, \lambda_p\}$ and $\{\mu_1, \ldots, \mu_q\}$ be the eigenvalues of the even operator A as above. Then consider the top coefficients a_p, b_q of the polynomials $P(z), Q(z)$ in (12). It follows that $a_p = \prod \lambda_i, b_q = \prod \mu_\alpha$, and $\operatorname{Ber} A = \frac{\prod \lambda_i}{\prod \mu_\alpha}$. Hence $\operatorname{Ber}^+(A) = R \cdot a_p$, $\operatorname{Ber}^-(A) = R \cdot b_q$, with a certain coefficient R. (Note that a_p and b_q are not polynomials in the matrix entries of A.) One can explicitly find a_p , b_q by solving straightforwardly a system of simultaneous equations corresponding to the linear recurrence relations (13). In particular, these calculations give

$$R = \det \begin{pmatrix} c_{p-q+1}(A) & \ldots & c_p(A) \\ \ldots & \ldots & \ldots \\ c_p(A) & \ldots & c_{p+q-1}(A) \end{pmatrix} .$$

By considering (14) one can come to an important observation that $R = \prod_{i,\alpha}(\lambda_i - \mu_a)$. Up to a sign it is just the classical Sylvester's resultant for the polynomials P and Q standing at the top and bottom of the characteristic function $R_A(z)$ in (12). Thus for the invariant polynomials $\operatorname{Ber}^+(A), \operatorname{Ber}^-(A)$ we have:

$$\operatorname{Ber}^+(A) = \prod_i \lambda_i \prod_{i,\alpha} (\lambda_i - \mu_\alpha) , \quad \operatorname{Ber}^-(A) = \prod_\alpha \mu_\alpha \prod_{i,\alpha} (\lambda_i - \mu_\alpha) .$$

The polynomials $\operatorname{Ber}^+(A), \operatorname{Ber}^-(A)$ appear in the analog of the Cayley–Hamilton theorem for the supercase. In particular, the polynomial

$$\mathcal{P}_A(z) = \operatorname{Ber}^+(A - z) \operatorname{Ber}^-(A - z) \cdot \frac{1}{R}$$

is the minimal annihilating polynomial for a generic even matrix A, and its coefficients are polynomial invariants of A.

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